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HIGHER-ORDER APPROXIMATIONS OF CNOIDAL-WAVE THEORY

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A solution of the problem of gravity waves on a liquid surface is sought in the form of a series whose first term corresponds to shallow-water theory. Such series have been previously studied numerically and analytically but their structure remains unclear because of the complicated initial formulation of the problem. In the present paper, instead of the strongly linear boundary-value problem with a free boundary containing several unknown functions, we solve an ordinary quadratic-nonlinear differential-difference equation of the first order containing an unknown function.

Introduction. We consider the classical problem of waves on a surface of a liquid of finite depth having one trough and one crest on a period. The liquid is an ideal and incompressible, and surface tension is absent. The bottom is even, the flow is steady, and the free surface is immovable and periodic.

In the present paper, when speaking of shallow-water theory, we mean the series expansion first described by K. Friedrichs and not the simple theory of shallow water without solitary waves. Below, this series is called a shallow-water series. Its first terms are found in the early works of Boussinesq, Rayleigh, Korteweg, and de Vries. The solutions obtained are called cnoidal waves because they contain the functions cn. K. Friedrichs proposed a systematic procedure for obtaining successive terms of the series, including different extensions in two different directions, introduction of the small parameter ε defined by these extensions, and an asymptotic series in ε.

The wave problem is appreciably simplified if it is considered not in the plane of physical variables but in the plane of a complex potential. In this case, instead of the problem in a region with a boundary that is not known in advance, we have a boundary-value problem in a known region — a band. In the present work, we seek a shallow-water series for the conformal mapping of the band onto the region occupied by the liquid. The solution should be periodic along the band since we seek periodic gravity waves. In [1–3], it is proposed that the solution should also be periodic across the band. This assumption proves useful because, in some cases, a solution that is periodic across the band can be found. The basis for this assumption is the fact that the shallow-water series contains the Jacobi elliptic functions cn (or sn), which are doubly periodic. Previously, this series has been examined only on the boundary and has not been studied outside of it. The above assumption can be validated by proving that all terms of the series are polynomials in the Jacobian functions.

Using a computer, Fenton obtained 9 terms of the series for solitary waves [4] and 5 terms for cnoidal waves [5]. In [6–8], respectively, 14, 17, and 27 terms of the series for solitary waves were obtained. Littman [9] and Ter-Krikorov [10] solved the problem in the plane of a complex potential and proved the existence of cnoidal waves.

Construction of higher-order approximations of shallow-water theory reduces to sequential search for

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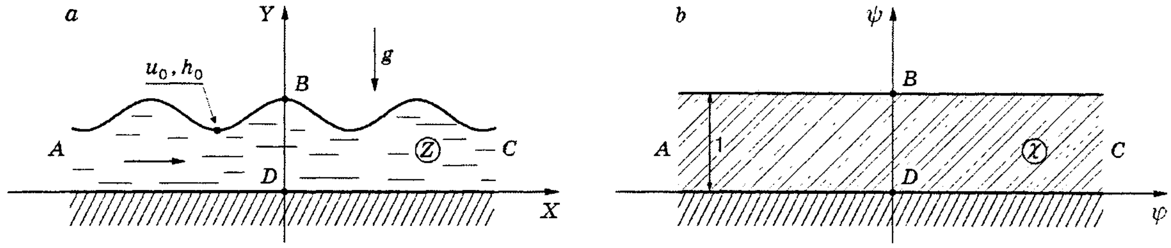

Fig. 1

terms of the series $g_j(z)$ from the differential equation

$$g_j'' + 4\rho^2[(1+k^2) - 3k^2\mathrm{sn}^2(\rho z, k)]g_j = h_j(z) \qquad (j \geqslant 2), \tag{1}$$

where ρ and k are free real parameters. The main difficulty is that of finding the right side of the equation $h_j(z)$. The conventional method of determining $h_j(z)$ from the terms found previously is cumbersome, and in the present work, explicit formulas for $h_j(z)$ are obtained. The functions $h_j(z)$ are even and periodic with period $2K(k)/\rho$. Here and below, $K(k)$ and $E(k)$ denote full elliptic integrals of the first and second order, respectively. The functions $g_j(z)$ should also be even and periodic. The following lemma, which is proved in [10], is valid.

Lemma 1. *If h_j is a continuous even periodic function with period $2K(k)/\rho$, then Eq.* (1) *has a unique even periodic solution $g_j(z)$ with the same period.*

However, the solution $g_j(z)$ is not unique since the function $h_j(z)$ is determined with accuracy up to an arbitrary additive constant μ_j. In the works cited above activities, this circumstance was not mentioned and the problem of choosing μ_j was not considered.

Formulation of the Problem. We place the origin of Cartesian coordinates at the bottom. The X axis is directed along the bottom from left to right, and the Y axis is directed vertically upward so that it passes through the wave crest (Fig. 1a).

Let h_0 and u_0 be the depth and velocity on the free surface in the wave trough and g be the acceleration of gravity. We assume that the streamfunction Ψ is equal to zero at the bottom and $\Psi = \Psi_0$ on the free surface. Let $\Psi_0 > 0$, i.e., the liquid moves from left to right.

Making the problem nondimensional gives rise to two constants: $\delta = \Psi_0/(u_0 h_0)$ and the Froude number $\mathrm{Fr} = u_0/\sqrt{gh_0}$. In the plane of the dimensionless complex potential $\chi = \varphi + i\psi = (\Phi + i\Psi)/\Psi_0$, the liquid corresponds to a band of width 1 (Fig. 1b). It is necessary to determine the conformal mapping of this band onto the region occupied by the liquid in the plane of physical variables. We seek this mapping in the form

$$Z = X + iY = h_0(2 + \mathrm{Fr}^2)f(\chi). \tag{2}$$

The choice of the coefficient $2 + \mathrm{Fr}^2$ here is not arbitrary. With this choice, the Bernoulli integral contains only one rather than two dimensionless constants:

$$\lambda = (\delta\,\mathrm{Fr})^2/(2 + \mathrm{Fr}^2)^3. \tag{3}$$

Let the point $\chi = 0$ be under the wave crest, i.e., $f(0) = 0$. The function $f(\chi)$ completely defines all flow parameters. Even without knowledge of Fr, examining $f(\varphi + i)$, we can determine the shape of the free surface in an unknown length scale. The Froude number Fr, and, hence, this scale are obtained from the condition $\mathrm{Im}\, f_{\mathrm{v}} = 1/(2 + \mathrm{Fr}^2)$, where f_{v} is the value of the conformal mapping at the lowest point on the free surface.

Thus, to find waves on water, we should solve the following problem.

Problem 1. To determine the constant λ and the function $f(\chi)$ that is analytic in the band

$$0 < \psi < 1, \qquad -\infty < \varphi < \infty \tag{4}$$

and satisfies the boundary conditions of constant pressure (Bernoulli integral)

$$\left|\frac{df}{d\chi}\right|^2 = \frac{\lambda}{1 - 2\,\mathrm{Im}\, f} \qquad (\psi = 1), \tag{5}$$

and even bottom

$$\mathrm{Im}\, f = 0 \qquad (\psi = 0). \tag{6}$$

It is believed that expansion in a shallow-water series should be preceded by different extensions in two different directions. For example, Friedrichs and Hyers [11], examining a particular case of solitary waves, write that if the horizontal independent variable is subjected to extension that depends on the parameter ε while the vertical independent variable is unchanged, the harmonic character of the functions describing the flow is disturbed. They believe that this procedure is obligatory with a direct approach to studying solitary waves. How does one construct the shallow-water series expansion for conformal mapping? If different extensions in φ and ψ are introduced, all advantages of the complex description of the problem are lost. We act differently: we assume that $f(\chi)$ is a function of χ that varies slowly, i.e., a small parameter ε exists and f depends on $z = x + iy = \varepsilon\chi$ and not on χ. The new function is denoted by the same character $f(z)$, and from the context below it is always clear which argument the function f has. The function $f(z)$ remains analytic, and its real and imaginary parts are harmonic functions.

Definition 1. A series expansion of the form

$$f(z) = \varepsilon^{-1} f_0(z) + \varepsilon f_1(z) + \varepsilon^3 f_2(z) + \dots \tag{7}$$

is called a shallow-water series.

We denote $g_j(z) = df_j(z)/dz$. Below, the derivative $df(z)/d\chi = g_0(z) + \varepsilon^2 g_1(z) + \varepsilon^4 g_2(z) + \dots$ is also called a shallow-water series expansion. At first glance, the Definition 1 is incomplete and should be supplemented by the series for the constant

$$\lambda = \lambda_0 + \varepsilon^2\lambda_1 + \varepsilon^4\lambda_2 + \dots \tag{8}$$

because in solving the Problem 1, it is necessary to use both series (7) and (8). Actually, series (7) is autonomous since, as shown below, there is a formulation of the wave problem that does not contain λ.

The shallow-water series is consistent if for $\varepsilon = 0$ there is a flow with a horizontal free surface and with the Froude number equal to unity, i.e., $\delta = \mathrm{Fr} = 1$. Thus, from (2) and (3) we determine the initial terms $f_0 = z/3$ and $\lambda_0 = 1/27$. The following terms of the series are found from the solution of the differential equations obtained by substitution of series (7) and (8) into (4)–(6) and equating terms with identical powers of ε. Since the calculations are rather complicated, we make a number of simplifications.

The first simplification consists of moving over from the boundary-value Problem 1 to an equation that is valid not only on the boundary but also in a certain region. By virtue of the symmetry principle, from the even bottom condition (6), we have $\overline{f(\varphi + i)} = f(\varphi - i)$. Hence, condition (5) can be written as

$$\frac{d f(\varphi + i)}{d\varphi}\,\frac{d f(\varphi - i)}{d\varphi} = \frac{\lambda}{1 + i[f(\varphi + i) - f(\varphi - i)]}. \tag{9}$$

We continue this equation analytically from the boundary. Replacing φ by χ, we obtain the cubic-nonlinear differential-difference equation

$$\frac{df(\chi + i)}{d\chi}\,\frac{df(\chi - i)}{d\chi} = \frac{\lambda}{1 + i[f(\chi + i) - f(\chi - i)]}. \tag{10}$$

However, the wave problem is quadratic-nonlinear, which was first shown by Babenko [12], who derived the corresponding operator equation. Therefore, the second simplification consists of obtaining a quadratic-nonlinear implication of (10). Replacing χ by $\chi + i$ and $\chi - i$, we obtain

$$\frac{df(\chi + 2i)}{d\chi}\,\{1 + i[f(\chi + 2i) - f(\chi)]\} = \frac{\lambda}{df(\chi)/d\chi}; \tag{11}$$

$$\frac{df(\chi - 2i)}{d\chi}\,\{1 + i[f(\chi) - f(\chi - 2i)]\} = \frac{\lambda}{df(\chi)/d\chi}. \tag{12}$$

Equating the left sides of (11) and (12), we obtain the required implication. Thus, the third simplification is achieved: the new equation does not contain λ. Thus, to find waves on water, instead of solving Problem 1, we solve the following problem.

Problem 2. To find an analytic function $f(\chi)$ that, in a certain region, satisfies the equation

$$\frac{df(\chi+2i)}{d\chi}\{1+i[f(\chi+2i)-f(\chi)]\} = \frac{df(\chi-2i)}{d\chi}\{1+i[f(\chi)-f(\chi-2i)]\}. \tag{13}$$

In transition from Problem 1 to Problem 2, the replacement of φ by χ, i.e., the conversion from (9) to (10), is important. This is equivalent to the analytic continuation of $f(\chi)$ from the boundary of the band (4) to a certain region outside the band, which is possible if there is a region without singular points outside the band (4). We assume, for example, that $f(\chi)$ is analytic in a certain rectangle $\varphi_1 < \varphi < \varphi_2$, $-2 < \psi < 3$. Then, Eq. (10) is valid in the smaller rectangle $\varphi_1 < \varphi < \varphi_2$, $-1 < \psi < 2$. The shift of the latter by $-i$ and $+i$ gives, accordingly, two new rectangles, in each of which (11) or (12) holds. The overlapping of these rectangles gives the rectangle $\varphi_1 < \varphi < \varphi_2$. $0 < \psi < 1$, i.e., the part of the band (4), in which (13) holds.

Solution of the Problem in Series. We cannot prove the analyticity of $f(\chi)$ outside the band (4), and, hence, we cannot prove that Problem 2 is an implication of Problem 1. However, it is important for us that the solutions in series (7) and (8) of Problems 1 and 2 give identical results.

Replacing χ by z in (13), we obtain

$$\frac{df(z+2i\varepsilon)}{dz}\{1+i[f(z+2i\varepsilon)-f(z)]\} = \frac{df(z-2i\varepsilon)}{dz}\{1+i[f(z)-f(z-2i\varepsilon)]\}. \tag{14}$$

Let us show that Eq. (14) can be written as $d\{\ldots\}/dz = 0$. Hence, the expression in braces should be a constant. We rewrite (14) in the form

$$i(A_1+A_2) + \frac{df(z+2i\varepsilon)}{dz} - \frac{df(z-2i\varepsilon)}{dz} = 0,$$

where

$$A_1 = \frac{1}{2}\frac{d}{dz}\{[f(z+2i\varepsilon)-f(z)]^2 + [f(z-2i\varepsilon)-f(z)]^2\},$$

$$A_2 = \frac{df(z)}{dz}\{f(z+2i\varepsilon)+f(z-2i\varepsilon)-2f(z)\}$$

and substitute the Taylor series expansion into these expressions:

$$f(z\pm 2i\varepsilon)-f(z) = \sum_{l=1}^{\infty}\frac{(\pm 2i\varepsilon)^l}{l!}\frac{d^l}{dz^l}f(z), \quad \frac{df(z\pm 2i\varepsilon)}{dz} = \sum_{l=1}^{\infty}\frac{(\pm 2i\varepsilon)^l}{l!}\frac{d^{l+1}}{dz^{l+1}}f(z).$$

The quantity A_1 is represented as a derivative. A similar representation can also be obtained for A_2 if we use the identity

$$2\frac{df(z)}{dz}\frac{d^{2j}f(z)}{dz^{2j}} = \frac{d}{dz}\sum_{p=1}^{2j-1}(-1)^{p+1}\frac{d^pf(z)}{dz^p}\frac{d^{2j-p}f(z)}{dz^{2j-p}}.$$

After some simplifications, we have

$$\frac{d}{dz}\Big\{i\sum_{m=1}^{\infty}(2i\varepsilon)^{2m}\sum_{p=1}^{2m-1}\gamma_{mp}\frac{d^pf(z)}{dz^p}\frac{d^{2m-p}f(z)}{dz^{2m-p}} + 2\sum_{m=0}^{\infty}\frac{(2i\varepsilon)^{2m+1}}{(2m+1)!}\frac{d^{2m+1}f(z)}{dz^{2m+1}}\Big\} = 0, \tag{15}$$

where $\gamma_{mp} = 1/(p!(2m-p)!) + (-1)^{p+1}/(2m)!$.

We substitute the shallow-water series expansion (7) into Eq. (15). Equating to zero the coefficients at ε^{2j} ($j \geqslant 1$), we obtain

$$\frac{d}{dz}\Big\{\sum_{m=1}^{j}(2i)^{2m}\sum_{p=1}^{2m-1}\gamma_{mp}\sum_{l=0}^{j-m}\frac{d^pf_l}{dz^p}\frac{d^{2m-p}f_{j-m-l}}{dz^{2m-p}} - \sum_{m=1}^{j}\frac{(2i)^{2m}}{(2m-1)!}\frac{d^{2m-1}f_{j-m}}{dz^{2m-1}}\Big\} = 0. \tag{16}$$

The first nontrivial equation of (16) for $j = 3$ has the form

$$\frac{d}{dz}\Big\{\frac{d^3f_1}{dz^3} + \frac{27}{2}\Big(\frac{df_1}{dz}\Big)^2\Big\} = 0.$$

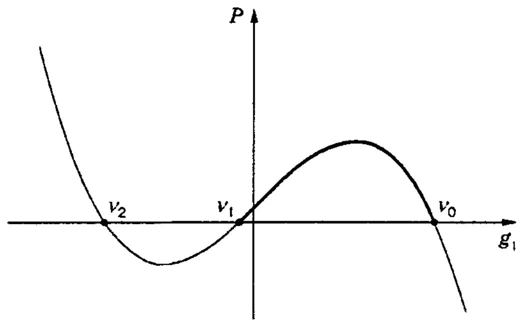

Fig. 2

The expression in braces should be a constant. We designate it by μ_1. Similar constants μ_{j-2} arise in integrating the next equations of (16). As a result, replacing $f_j(z)$ by $g_j(z)$, we formulate the following theorem.

Theorem 1. *The functions $g_j(z)$ satisfy the differential equations*

$$g_1'' + (27/2)g_1^2 = \mu_1, \tag{17}$$

$$g_j'' + 27g_1g_j = h_j \qquad (j \geqslant 2), \tag{18}$$

where

$$h_j = \sum_{m=2}^{j}\sum_{p=1}^{2m-1} \alpha_{mp} \sum_{l=1}^{j-m+1} \frac{d^{p-1}g_l}{dz^{p-1}} \frac{d^{2m-p-1}g_{j+2-m-l}}{dz^{2m-p-1}}$$

$$-\frac{27}{2}\sum_{l=2}^{j-1} g_l g_{j+1-l} + \sum_{l=3}^{j+1}\beta_l \frac{d^{2l-2}g_{j+2-l}}{dz^{2l-2}} + \mu_j. \tag{19}$$

The constants α_{mp} and β_l are obtained from the formulas

$$\alpha_{mp} = 9(-1)^m 2^{2(m-1)}\Big[\frac{1}{p!(2m-p)!} + \frac{(-1)^{p+1}}{(2m)!}\Big], \quad \beta_l = 3(-1)^l 2^{2l-1}\frac{1-l}{(2l)!},$$

and the constants μ_j are not determined.

Thus we have a recurrent chain of differential equations from which $g_j(z)$ can be sequentially obtained. Below, it is shown that there is a set of even periodic solutions of Eqs. (17) and (18) that depend on two real parameters. The unknown μ_1 is expressed in terms of these parameters. The remaining constants μ_j $(j \geqslant 2)$ are arbitrary numbers, and an even periodic solution $g_j(z)$ exists regardless of how they are chosen.

First-Order Approximation. Integrating Eq. (17), we obtain

$$(g_1')^2 = -9g_1^3 + 2\mu_1 g_1 + \text{const} = P(g_1). \tag{20}$$

The cubic polynomial $P(g_1) = -9(g_1 - \nu_0)(g_1 - \nu_1)(g_1 - \nu_2)$ should have only real roots ν_0, ν_1, and ν_2. Indeed, the velocity at the bottom under the crest and the trough reaches an extremum. Since the quantity $\text{const}/(df/d\chi)$ has the meaning of velocity, g_1' should be equal to zero at these points. From (20) we find that $P(g_1) = 0$ at these two points at the bottom. Thus, at least two roots $P(g_1)$ are real since the function g_1 is real everywhere at the bottom. The reality of the third root follows from the reality of the polynomial coefficients.

Without loss of generality, we set $\nu_0 > \nu_1 > \nu_2$. The coefficient at g_1^2 in $P(g_1)$ is equal to zero, and, hence,

$$\nu_0 + \nu_1 + \nu_2 = 0. \tag{21}$$

We examine Eq. (20) at the bottom, i.e., we set $z = x$. Under the wave crest $(x = 0)$, the liquid velocity has a minimum. Therefore, with increase in x, we have $g_1' < 0$ in the neighborhood to the right of the point $x =$

0. This allows us to select the proper sign in extracting the root: $dg_1/dx = -3\sqrt{-(g_1-\nu_0)(g_1-\nu_1)(g_1-\nu_2)}$ $(x > 0)$. The radicand should be positive and limited. On the plot of $P(g_1)$, these conditions correspond to the chosen segment (Fig. 2). The function g_1 takes the maximum value ν_0 under the crest $x = 0$. Therefore, the constant of integration is known:

$$\int\limits_{\nu_0}^{g_1} \frac{d\xi}{\sqrt{-(\xi-\nu_0)(\xi-\nu_1)(\xi-\nu_2)}} = -3x.$$

On the left side there is an elliptic integral (for more details see [13]), which is inverted to give

$$g_1 = \nu_0 - (\nu_0 - \nu_1)\,\mathrm{sn}^2(\rho x, k) \qquad (0 \leqslant k \leqslant 1); \tag{22}$$

$$\rho = \frac{3}{2}\sqrt{\nu_0 - \nu_2}, \qquad k = \sqrt{\frac{\nu_0-\nu_1}{\nu_0-\nu_2}}. \tag{23}$$

The five parameters ν_0, ν_1, ν_2, ρ, and k are related by three Eqs. (21) and (23). Therefore, two parameters — ρ and the modulus of the elliptic functions k — can be considered independent.[1]

Expressing all parameters in (22) in terms of ρ and k and replacing x by z, we find the first nontrivial term of the shallow-water series expansion:

$$g_1 = \rho^2\Big[\frac{4}{27}(1+k^2) - \frac{4}{9}k^2\,\mathrm{sn}^2\rho z\Big], \quad f_1 = \rho^2\Big[\frac{4}{27}(1+k^2)z - \frac{4}{9}k^2\int\limits_0^z \mathrm{sn}^2\rho\xi\,d\xi\Big].$$

To obtain a solution expressed in terms of χ, it is necessary to introduce the new small parameter $\theta = \varepsilon\rho$. The conformal mapping then takes the form

$$f(\chi) = \frac{1}{3}\chi + \theta^2\Big\{\frac{4}{27}(1+k^2)\chi - \frac{4}{9}k^2\int\limits_0^\chi \mathrm{sn}^2\theta\zeta\,d\zeta\Big\} + O(\theta^4).$$

We obtained a two-parameter set of wave solutions, with the wave amplitude and length depending on both parameters θ and k $(0 \leqslant k \leqslant 1)$. As $k \to 0$ and $k \to 1$, we have sn $\to$ sin and sn $\to$ th. These limiting cases correspond to sinusoidal and solitary waves:

$$f(\chi) \simeq \chi\Big(\frac{1}{3} + \theta^2\frac{4}{27}\Big) + \theta k^2\frac{1}{9}\sin 2\theta\chi, \quad f(\chi) \simeq \chi\Big(\frac{1}{3} - \theta^2\frac{4}{27}\Big) + \theta\frac{4}{9}\tanh\theta\chi.$$

Second-Order Approximation. We now solve of Eqs. (18). We introduce designations for the arguments of the elliptic functions $u = \rho z$ and for the Jacobi elliptic functions $\mathrm{s} = \mathrm{sn}\,u$, $\mathrm{c} = \mathrm{cn}\,u$ and $\mathrm{d} = \mathrm{dn}\,u$.

We consider the homogeneous equation (18)

$$g_j'' + 27 g_1 g_j = 0. \tag{24}$$

Differentiating (17), we see that g_1' is a solution of this equation. Thus, we obtain the first solution of (24):

$$v(z) = \frac{d\,\mathrm{s}^2}{dz} = 2\rho\,\mathrm{scd}. \tag{25}$$

Because the Wronskian is equal to unity, the second linearly independent solution is $v\int dz/v^2$. Hence, the general solution (18) can be written as

$$g_j = \tilde{c}_1 v + \tilde{c}_2 v\int\frac{dz}{v^2} + v\int\frac{dz}{v^2}\int v h_j\,dz. \tag{26}$$

[1]Without loss of generality, the parameter ρ can be set equal to unity because, as shown below, it is contained only in the product $\varepsilon\rho$. Treating the product $\varepsilon\rho$ as a new small parameter is equivalent to the assumption that $\rho = 1$.

If the function h_j is known, then, according to Lemma 1 for the even and periodic function $g_j(z)$, the constants $\tilde{c}_1$ and $\tilde{c}_2$ should be determined uniquely.

We calculate g_2. From (19) for $j = 2$, we have

$$h_2 = 15 g_1 \frac{d^2 g_1}{dz^2} + \frac{15}{2}\Big(\frac{dg_1}{dz}\Big)^2 + \frac{4}{15}\frac{d^4 g_1}{dz^4} + \mu_2. \tag{27}$$

The quantity h_2 is a polynomial in s^2. To prove this statement, we use the following lemma.

Lemma 2. *If G_1 and G_2 are polynomials in s^2 of degrees n_1 and n_2, respectively, the product*

$$\frac{d^{2p-j}G_1}{dz^{2p-j}}\frac{d^jG_2}{dz^j} \qquad (0 \leqslant j \leqslant 2p)$$

$$\frac{d^{2p-j}G_1}{dz^{2p-j}}\frac{d^jG_2}{dz^j} \qquad (0 \leqslant j \leqslant 2p)$$

is a polynomial in s^2 of degree $n_1 + n_2 + p$.

Proof. By induction over j invoking the formula $d/dz = 2\rho\sqrt{(1-\mathrm{s}^2)(1-k^2\mathrm{s}^2)}\,d/d\mathrm{s}$, we find that if G is a polynomial in s^2 of degree n, then its odd and even derivatives are determined from the formulas

$$\frac{d^{2j-1}G}{dz^{2j-1}} = \mathrm{s}\sqrt{(1-\mathrm{s}^2)(1-k^2\mathrm{s}^2)}\,M, \qquad \frac{d^{2j}G}{dz^{2j}} = N,$$

where M and N are polynomials in s^2 of degrees $n+j-2$ and $n+j$, respectively. Thus, Lemma 2 is proved.

Lemma 2 can be written in simplified form: each differentiation of a polynomial in s with respect to z increases its degree by unity. Because g_1 is a polynomial in s^2 of degree 1, then from (27) it follows that h_2 is a polynomial in s^2 of degree 3:

$$h_2 = b_2^0 + b_2^1\mathrm{s}^2 + b_2^2\mathrm{s}^4 + b_2^3\mathrm{s}^6. \tag{28}$$

We note that all coefficients of this polynomial are known, except for b_2^0, because this coefficient contains the unknown constant μ_2.

In view of (25) and (28), we have $\int v h_2\,dz = \int h_2\,d(\mathrm{s}^2) = b_2^0\mathrm{s}^2 + \frac{1}{2}b_2^1\mathrm{s}^4 + \frac{1}{3}b_2^2\mathrm{s}^6 + \frac{1}{4}b_2^3\mathrm{s}^8$. Denoting $J_n = \mathrm{scd}\int \frac{\mathrm{s}^{2n-4}}{\mathrm{c}^2\mathrm{d}^2}\,du$, from (26) we obtain

$$g_2 = c_1\,\mathrm{scd} + c_2 J_1 + \frac{1}{2\rho^2}\Big(b_2^0 J_2 + \frac{1}{2}b_2^1 J_3 + \frac{1}{3}b_2^2 J_4 + \frac{1}{4}b_2^3 J_5\Big). \tag{29}$$

All integrals J_n $(1 \leqslant n \leqslant 5)$ included in (29) can be calculated. They consist of three terms: a square polynomial in s^2 and the functions $u\,\mathrm{scd}$ and $\mathrm{scd}\int \mathrm{d}^2\,du$, multiplied by certain constants, and have the form $J_n = D_n^0 + D_n^1\mathrm{s}^2 + D_n^2\mathrm{s}^4 + E_n u\,\mathrm{scd} + F_n\,\mathrm{scd}\int \mathrm{d}^2\,du$. This can be proved by direct differentiation using, for example, the MAPLE system. The constants D_n^m, E_n, and F_n are shown in Table 1.

From (29) it follows that g_2 also consists of a polynomial in s^2 and the functions $u\,\mathrm{csd}$ and $\mathrm{scd}\int \mathrm{d}^2\,du$:

$$g_2 = a_2^0 + a_2^1\mathrm{s}^2 + a_2^2\mathrm{s}^4 + c_1\,\mathrm{scd} + \mathrm{scd}\Big(\omega_1 u + \omega_2\int_0^u \mathrm{d}^2\,du\Big). \tag{30}$$

In (29), among the constants c_1, c_2, b_2^0, b_2^1, b_2^2, and b_2^3, the unknowns to be determined are c_1, c_2, and b_2^0. In (30), the unknown constant c_1 is retained, and the new constants — the polynomial coefficients a_2^0, a_2^1, and a_2^2 and ω_1 and ω_2 — are functions of the unknowns c_2 and b_2^0. For example, in ω_1 and ω_2, the unknowns enter as follows:

$$\omega_1 = \frac{1}{k^2-1}\Big\{c_2(k^2-2) - \frac{b_2^0}{2\rho^2} + \ldots\Big\}, \tag{31}$$

TABLE 1

n	D_n^0	D_n^1	D_n^2	E_n	F_n
1	-1	$\frac{(1+k^2)(2k^4-3k^2+2)}{(k^2-1)^2}$	$-\frac{2k^2(1-k^2+k^4)}{(k^2-1)^2}$	$\frac{k^2-2}{k^2-1}$	$-\frac{2(1-k^2+k^4)}{(k^2-1)^2}$
2	0	$\frac{k^4+1}{(k^2-1)^2}$	$-\frac{k^2(1+k^2)}{(k^2-1)^2}$	$-\frac{1}{k^2-1}$	$-\frac{1+k^2}{(k^2-1)^2}$
3	0	$\frac{k^2+1}{(k^2-1)^2}$	$-\frac{2k^2}{(k^2-1)^2}$	$-\frac{1}{k^2-1}$	$-\frac{2}{(k^2-1)^2}$
4	0	$\frac{2}{(k^2-1)^2}$	$-\frac{1+k^2}{(k^2-1)^2}$	$-\frac{1}{k^2(k^2-1)}$	$-\frac{1+k^2}{k^2(k^2-1)^2}$
5	0	$\frac{1+k^2}{k^2(k^2-1)^2}$	$-\frac{1+k^4}{k^2(k^2-1)^2}$	$\frac{k^2-2}{k^4(k^2-1)}$	$-\frac{2(k^4-k^2+1)}{k^4(k^2-1)^2}$

$$\omega_2 = -\frac{1}{(k^2-1)^2}\Big\{2c_2(1-k^2+k^4) + \frac{b_2^0}{2\rho^2}(1+k^2) + \ldots\Big\}.$$

Let the constant μ_2 on the right side (27) be known. Then, one of the three unknowns b_2^0 is specified. Hence, the function h_2 is determined, and according to Lemma 1, the unknowns c_1 and c_2 are determined uniquely. It is necessary to set $c_1 = 0$ because the periodic function scd is odd. Next, if we require that the quantity $\omega_1 u + \omega_2 \int_0^u \mathrm{d}^2\, du$ be periodic, then, as follows from (30), the function g_2 is both even and periodic. This requirement is satisfied because although the function $\int_0^u \mathrm{d}^2\, du$ is not periodic, its value increases by $2K(k)$ as u increases by $2E(k)$. Hence, g_2 is a periodic function if

$$\omega_1 K(k) + \omega_2 E(k) = 0. \tag{32}$$

Substituting (31) into (32), we obtain an equation for the unknown c_2.

However, the constant μ_2 is unknown, and, hence, the function g_2 is not determined uniquely, generally speaking. Let us show that the nonuniqueness is related to the uncertainty of the parameter ε. We analyze how the shallow-water series expansion

$$\frac{df}{d\chi} = \frac{1}{3} + \varepsilon^2 g_1(z) + \varepsilon^4 g_2(z) + \ldots = \frac{1}{3} + \varepsilon^2 g_1(\varepsilon\chi) + \varepsilon^4 g_2(\varepsilon\chi) + \ldots$$

changes when ε is replaced by $\varepsilon + a\varepsilon^3 + \ldots$, where a is a real number. With allowance for the Taylor series $g_1([\varepsilon + a\varepsilon^3 + \ldots]\chi) = g_1(\varepsilon\chi) + a\varepsilon^3\chi\, dg_1/dz + \ldots$, we obtain the new shallow-water series expansion $df/d\chi = 1/3 + \varepsilon^2 g_1(z) + \varepsilon^4 \tilde{g}_2(z) + \ldots$, where $\tilde{g}_2(z) = g_2(z) + 2ag_1(z) + az\, dg_1(z)/dz$. Hence, the above change in ε leads to a change in $g_2(z)$. Because $z\, dg_1/dz$ coincides with u scd with accuracy to a numerical coefficient, the last statement can be refined: with a change in ε the coefficient ω_1 in (30) changes. It is possible to select ε such that $\omega_1 = 0$, and, hence, it is also necessary to set $\omega_2 = 0$ (otherwise, g_2 is a nonperiodic function). Only in this case is the function g_2 a polynomial in s^2. The quantity a is determined uniquely from the equation $\omega_1 = 0$.

Thus, there is a parameter ε for which g_2 has the form of a square polynomial in s^2: $g_2 = a_2^0 + a_2^1\mathrm{s}^2 + a_2^2\mathrm{s}^4$. The coefficients of this polynomial are expressed in terms of the unknowns c_2 and b_2^0 ($c_1 = 0$), which can be found from the solution of the linear system $\omega_1 = 0$, $\omega_2 = 0$ [ω_1 and ω_2 are determined in (31)]. As a result, we have

$$g_2 = \rho^4[-(16/1215)(13k^4 - 43k^2 + 13) - (64/81)k^2(1+k^2)\mathrm{s}^2 + (32/27)k^4\mathrm{s}^4].$$

Higher-Order Approximation. We prove by induction over j that g_j are polynomials in s^2. Let the inductive hypothesis be valid: for $l < j$, all g_l are polynomials in s^2 of degree l. We solve Eqs. (18). Lemma 2 leads directly to

Lemma 3. *If* $g_l = a_l^0 + a_l^1 s^2 + \ldots + a_l^l s^{2l}$ $(l < j)$, then the function h_j is a polynomial in s^2 of order $j+1$: $h_j = b_j^0 + b_j^1 s^2 + \ldots + b_j^{j+1} s^{2(j+1)}$.

Taking into account (25), from (26) we obtain the following formula, which is similar to (29):

$$g_j = c_1 \,\mathrm{scd} + c_2 J_1 + \frac{1}{2\rho^2} \sum_{n=2}^{j+3} \frac{J_n b_j^{n-2}}{n-1} \qquad (j > 2). \tag{33}$$

Here the unknowns are c_1, c_2, and b_j^0. Previously, the form of the integrals J_n was obtained only for $1 \leqslant n \leqslant 5$. To establish the form of the remaining integrals which are included in (33), we need

Lemma 4. *For* $n \geqslant 5$, *we have*

$$J_n = P_n(\mathrm{s}) + Q_n u \,\mathrm{scd} + R_n \,\mathrm{scd} \int \mathrm{d}^2 \, du, \tag{34}$$

where $P_n(\mathrm{s})$ *is a polynomial in* s^2 *of degree* $n-3$ *and* Q_n *and* R_n *are constants.*

Proof. We designate $K_n = \mathrm{scd} \int \mathrm{s}^{2n} \, du$. Using the recursive formula

$$K_n = \frac{1}{(2n-1)k^2} \mathrm{c}^2 \mathrm{d}^2 \mathrm{s}^{2n-2} + \frac{(2n-2)(1+k^2)}{(2n-1)k^2} K_{n-1} - \frac{2n-3}{(2n-1)k^2} K_{n-2},$$

it is not difficult to prove by induction over n that

$$K_n = L_n(\mathrm{s}) + M_n u \,\mathrm{scd} + N_n \,\mathrm{scd} \int \mathrm{d}^2 \, du, \tag{35}$$

where $L_n(\mathrm{s})$ is a polynomial in s^2 of order $n+1$ and M_n and N_n are constants. Using (35) and applying induction over n to

$$J_{n+2} = \frac{1}{k^2} J_{n+1} + \frac{1}{k^2} \sum_{j=0}^{n-2} K_j - \frac{\mathrm{d}^2 \mathrm{c}^2}{k^2(1-k^2)} - \frac{u \,\mathrm{scd}}{k^2} + \frac{\mathrm{scd}}{k^2(1-k^2)} \int \mathrm{d}^2 \, du,$$

we obtain (34). Lemma 4 is proved.

From (33) we have

$$g_j = \sum_{n=0}^{j} a_j^n \mathrm{s}^{2n} + c_1 \,\mathrm{scd} + \mathrm{scd}\left(\omega_1 u + \omega_2 \int_0^u \mathrm{d}^2 \, du \right). \tag{36}$$

The further reasoning is similar to the one above. In (36), we set $c_1 = 0$ and express the remaining constants a_j^n, ω_1, and ω_2 in terms of c_2 and b_j^0. For example, ω_1 and ω_2 are determined from formulas that follow from (31) when b_2^0 is replaced by b_j^0. Replacing ε by $\varepsilon + a\varepsilon^{2j-1} + \ldots$, we replace $g_j(z)$ by $g_j(z) + 2a g_1(z) + az \, dg_1/dz$, and, hence, change ω_1 in (36). We set $\omega_1 = 0$. This condition determines the constant a uniquely. Requiring that $\omega_2 = 0$, we eliminate the nonperiodic term. Next, solving the system $\omega_1 = 0$, $\omega_2 = 0$, we obtain c_2 and b_j^0 and then, the polynomial coefficients a_j^n. Thus, by induction over j, we proved the following theorem.

Theorem 2. *There exists a unique parameter* ε *such that in the shallow-water series expansion*

$$\frac{df}{d\chi} = \frac{1}{3} + \varepsilon^2 g_1(\varepsilon\chi) + \varepsilon^4 g_2(\varepsilon\chi) + \ldots \tag{37}$$

each term g_j *is a polynomial in* s^2 *of degree* j, *i.e.*,

$$g_j = a_j^0 + a_j^1 \mathrm{s}^2 + \ldots + a_j^j \mathrm{s}^{2j}. \tag{38}$$

Computations. We seek g_j in the form of the polynomial (38). Substituting (38) into Eq. (18) and equating terms with identical powers of s, we have a linear system of equations with a triangular matrix. The five initial terms of the series (37) are

$$\frac{g_1}{\rho^2} = -\frac{4}{9} k^2 \mathrm{s}^2 + \frac{4}{27}(1+k^2), \quad \frac{g_2}{\rho^4} = \frac{32}{27} k^4 \mathrm{s}^4 - \frac{64}{81} k^2 (1+k^2) \mathrm{s}^2 - \frac{16}{1215}(13 - 43k^2 + 13k^4),$$

$$\frac{g_3}{\rho^6} = -\frac{4352}{1215} k^6 \mathrm{s}^6 + \frac{4352}{1215} k^4 (1+k^2) \mathrm{s}^4 - \frac{256}{3645} k^2 (2 + 49k^2 + 2k^4) \mathrm{s}^2$$

$$+\frac{64}{76{,}545}(1+k^2)(242\,k^4-521\,k^2+242),$$

$$\frac{g_4}{\rho^8}=\frac{1{,}040{,}384}{91{,}125}\,k^8\mathrm{s}^8-\frac{4{,}161{,}536}{273{,}375}\,k^6(1+k^2)\mathrm{s}^6$$

$$+\frac{16{,}384}{273{,}375}\,k^4(59\,k^4+322\,k^2+59)\mathrm{s}^4-\frac{1024}{5{,}740{,}875}\,k^2(1+k^2)(1898\,k^4+15{,}079\,k^2+1898)\mathrm{s}^2$$

$$-\frac{256}{17{,}222{,}625},(16{,}135-43{,}658\,k^2+112{,}101\,k^4-43{,}658\,k^6+16{,}135\,k^8),$$

$$\frac{g_5}{\rho^{10}}=-\frac{11{,}165{,}696}{297{,}675}\,k^{10}\mathrm{s}^{10}+\frac{11{,}165{,}696}{178{,}605}\,k^8(1+k^2)\mathrm{s}^8-\frac{16{,}384}{120{,}558{,}375}\,k^6(189{,}041+730{,}984\,k^2$$

$$+189{,}041\,k^4)\mathrm{s}^6+\frac{8192}{120{,}558{,}375}\,k^4(1+k^2)(37{,}528\,k^4 473{,}303\,k^2+37{,}528)\mathrm{s}^4+$$

$$+\frac{8192}{361{,}675{,}125}\,k^2(26{,}909-166{,}402\,k^2-111{,}600\,k^4-166{,}402\,k^6+26{,}909\,k^8)\mathrm{s}^2$$

$$+\frac{1024}{11{,}935{,}279{,}125}(1+k^2)(3{,}314{,}710\,k^8-15{,}153{,}473\,k^6+7{,}595{,}607\,k^4-15{,}153{,}473\,k^2+3{,}314{,}710).$$

The above formulas are rather complicated. They are polynomials in s^2 whose coefficients are, in turn, polynomials in k^2. We note that this is not a single representation of the solution. It is possible to represent g_j as polynomials in c^2 or d^2 whose coefficients are also polynomials in k^2. The simplest representation of the solution is obviously obtained if g_j is expressed via the first term of the series g_1. Introducing the new variable $\zeta=-(4/9)k^2\mathrm{s}^2+(4/27)(1+k^2)$, we write g_j in the form of polynomials in ζ:

$$\frac{g_1}{\rho^2}=\zeta,\qquad \frac{g_2}{\rho^4}=6\,\zeta^2-\frac{368}{1215}(1-k^2+k^4),$$

$$\frac{g_3}{\rho^6}=\frac{204}{5}\,\zeta^3-\frac{64}{27}(1-k^2+k^4)\zeta+\frac{9664}{45{,}927}(k^2-2)(-1+2\,k^2)(1+k^2),$$

$$\frac{g_4}{\rho^8}=\frac{36{,}576}{125}\,\zeta^4-\frac{69{,}632}{3375}(1-k^2+k^4)\zeta^2$$

$$+\frac{2{,}936{,}576}{1{,}913{,}625}(k^2-2)(-1+2\,k^2)(1+k^2)\zeta-\frac{2{,}196{,}736}{5{,}740{,}875}(1-k^2+k^4)^2,$$

$$\frac{g_5}{\rho^{10}}=\frac{2{,}649{,}672}{1225}\,\zeta^5-\frac{30{,}114{,}304}{165{,}375}(1-k^2+k^4)\zeta^3+\frac{1{,}944{,}064}{165{,}375}(k^2-2)(-1+2k^2)(1+k^2)\zeta^2$$

$$-\frac{20{,}860{,}928}{13{,}395{,}375}(1-k^2+k^4)^2\zeta+\frac{7{,}820{,}391{,}424}{35{,}805{,}837{,}375}(k^2-2)(-1+2k^2)(1+k^2)(1-k^2+k^4).$$

The new formulas are much simpler. The coefficients of ζ are polynomials in k^2, which can be factored, and only the numerical coefficient remains undetermined. There are two types of factors: the expression $1-k^2+k^4$ or this expression multiplied by $(k^2-2)(2k^2-1)(k^2+1)$.

We emphasize the property $g_j(k^2,\zeta)=(-1)^j g_j(1-k^2,-\zeta)$, which implies that the solution expressed in terms of θ^2, k^2, and ζ has invariance:

$$\frac{df}{d\chi}(\theta^2,k^2,\zeta)=\frac{df}{d\chi}(-\theta^2,1-k^2,-\zeta).$$

This property appears to be important since it gives an implicit nonlinear transformation that relates two solutions with different wavelength. For example, solitary waves ($k = 1$) and waves on a surface of an infinitely deep liquid ($k = 0$) are related by the transformation $\theta \to i\theta$.

Conclusion. In the present paper, we propose a new method for deriving the shallow-water expansion based on replacing the integrodifferential equation to which the wave problem is usually reduced by the differential-difference equation (13). As compared to the existing methods, the proposed method is simpler and can be useful, in particular, in numerical calculations since it allows one to construct series more rapidly and more precisely and to find a greater number of series terms. Other studies (see, for example, [6–8]) employ a more complex method, in which four series, whose terms are functions of two variables, are used simultaneously.

In the theoretical aspect, the proposed method is also of interest:

1) for Eq. (13), exact solutions are known [1–3];

2) the Stokes series expansion (the second known expansion of the theory of waves on water) is obtained in a natural fashion from (13) if a solution of this equation is sought in the form of the series

$$f(\chi) = \tilde{f}_0(\chi) + \varepsilon\tilde{f}_1(\chi) + \varepsilon^2\tilde{f}_2(\chi) + \ldots . \qquad (39)$$

It is known that the Stokes series expansion is not adequate for describing long waves, and, in contrast, the shallow-water series expansion is appropriate for long waves but unsuitable for short waves. It is now possible to compare both series (7) and (39) and to attempt to construct a series expansion that applies universally for all wavelengths.

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